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NORM ATTAINING OPERATORS

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ABSTRACT
Every Banach space is isomorphic to a space with the property that the norm-attaining operators are dense in the space of all operators into it, for any given domain space. A super-reflexive space is arbitrarily nearly isometric to a space with this property.

A bounded linear operator $T : X \to Y$ between two Banach spaces is said to be *norm attaining* if there is an $x \in X$ with $\|x\| = 1$ and $\|Tx\| = \|T\|$. We write $\mathrm{NA}(X, Y)$ to denote the class of norm-attaining operators from X to Y. Bishop and Phelps [1] proved that if Y is one-dimensional then $\mathrm{NA}(X, Y)$ is norm dense in $X^* = B(X, Y)$ for all spaces X and raised the general question of when $\mathrm{NA}(X, Y)$ is dense in $B(X, Y)$.

Lindenstrauss [6] defined the following two properties of a Banach space. X has *property A* if $\mathrm{NA}(X, Y)$ is dense in $B(X, Y)$ for all spaces Y. Y has *property B* if $\mathrm{NA}(X, Y)$ is dense in $B(X, Y)$ for all spaces X. Whereas much is known about property A (see, for example, Bourgain [2] and Diestel and Uhl [4]), considerably less is known about property B. For instance, it is known that l_∞ and c_0 have B, but it is unknown even whether l_1 and l_2 do. Property B is not an isomorphism invariant; for example, c_0 with an equivalent strictly convex norm lacks it [6].

Johnson and Wolfe [5] raised the question whether every Banach space could be equivalently renormed to have property B. In this note we shall settle the question affirmatively (Theorem 1). Lindenstrauss gave the following sufficient condition for a space to have property B.

LEMMA 1 [6]. *Let X be a Banach space such that there exists a set $\{x_\alpha, f_\alpha : \alpha \in A\}$ with $x_\alpha \in X$, $f_\alpha \in X^*$ and $\lambda < 1$ such that*

(i) $\|f_\alpha\| = 1$ *for each* α *and* $\|x\| = \sup_\alpha |f_\alpha(x)|$ *for all* $x \in X$;

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(ii) $\|x_\alpha\| = f_\alpha(x_\alpha) = 1$ *for each* α *and* $|f_\alpha(x_\beta)| < \lambda$ *for* $\alpha \neq \beta$.
Then X *has property* B.

Typical examples of spaces satisfying the above conditions are l_∞, c_0 and finite-dimensional spaces with polyhedral unit balls. In the following we will use Greek letters to designate ordinal numbers and identify cardinal numbers with the corresponding initial ordinals.

THEOREM 1. *Let* $(X, \|.\|)$ *be a Banach space. Then for any* $K > 3$ *there is a norm* $|||.|||$ *on* X *such that* $\|x\| \leqq |||x||| \leqq K\|x\|$, *and* $(X, |||.|||)$ *has property* B.

PROOF. Let γ be the density character of X: that is, there is a set $\{u_\alpha : \alpha < \gamma\}$ with $\|u_\alpha\| = 1$ for each α, which is norm dense in the unit sphere of X and γ is the least ordinal with this property. We may select $\{u^*_\alpha : \alpha < \gamma\} \subset X^*$ such that $\|u^*_\alpha\| = u^*_\alpha(u_\alpha) = 1$ for each α. Let s be a constant with $1 > s > 2/(K-1)$. By transfinite induction we may for each $\alpha < \gamma$ find in turn $y^*_\alpha \in X^*$ and $y_\alpha \in X$ with $\|y^*_\alpha\| = 1$ and $y^*_\alpha(y_\beta) = y^*_\alpha(u_\beta) = 0$ for all $\beta < \alpha$ (since the linear span of $\{y_\beta, u_\beta : \beta < \alpha\}$ cannot be dense) and such that $\|y_\alpha\| = 1$ and $y^*_\alpha(y_\alpha) > s$.

Let M be a constant with $K - 1 > M > 2/s$. Since $(u^*_\alpha + My^*_\alpha)(u_\alpha) + (u^*_\alpha - My^*_\alpha)(u_\alpha) = 2$, we may select a set of signs $\varepsilon_\alpha = \pm 1$ such that $g_\alpha = u^*_\alpha + \varepsilon_\alpha M y^*_\alpha$ satisfies $|g_\alpha(u_\alpha)| \geqq 1$ for all α. We choose constants r and D such that

$$1 > r > (M+1)/K \quad \text{and} \quad D > (2 + M(1-s))/(1-r).$$

We may now select a subset A of γ and a set of modulus one scalars, $\{\delta_\alpha : \alpha \in A\}$ by transfinite induction, satisfying the condition that $0 \in A$ and, for $\alpha > 0$, $\alpha \in A$ if and only if $|g_\beta(u_\alpha)| \leqq r$ for all $\beta \in A$ with $\beta < \alpha$. We choose δ_α such that ε_α and $\delta_\alpha g_\alpha(u_\alpha)$ have the same sign. Let $z_\alpha = y_\alpha + \delta_\alpha D u_\alpha$.

Now if $\alpha, \beta \in A$ and $\beta < \alpha$,

$$|g_\alpha(z_\alpha)| = |u^*_\alpha(y_\alpha) + \varepsilon_\alpha M y^*_\alpha(y_\alpha) + \delta_\alpha D g_\alpha(u_\alpha)| \geqq Ms + D - 1$$

since $y^*_\alpha(y_\alpha) > s$, $|g_\alpha(u_\alpha)| \geqq 1$ and ε_α and $\delta_\alpha g_\alpha(u_\alpha)$ have the same sign;

$$|g_\beta(z_\alpha)| = |u^*_\beta(y_\alpha) + \varepsilon_\beta M y^*_\beta(y_\alpha) + \delta_\alpha D g_\beta(u_\alpha)| \leqq 1 + M + rD$$

since $|g_\beta(u_\alpha)| \leqq r$;

$$|g_\alpha(z_\beta)| = |u^*_\alpha(y_\beta + \delta_\beta D u_\beta)| \leqq 1 + D,$$

since $y^*_\alpha(y_\beta) = y^*_\alpha(u_\beta) = 0$.

Moreover, $Ms + D - 1 > 1 + M + rD$ by the choice of D, and $Ms + D - 1 > 1 + D$ by the choice of M.

Let $f_\alpha = g_\alpha / r$ for $\alpha \in A$, and define $|||x|||$ to be $\sup_{\alpha \in A} |f_\alpha(x)|$ for $x \in X$. Then $|||x||| \leqq (1+M)\|x\|/r$ and, given $\alpha < \gamma$, either $\alpha \in A$, so that $|||u_\alpha||| \geqq 1/r$, or else $\alpha \notin A$, so that

$$|f_\beta(u_\alpha)| > 1 \qquad \text{for some } \beta \in A,\ \beta < \alpha.$$

Since $\{u_\alpha : \alpha < \gamma\}$ is dense on the unit sphere of X, $|||x||| \geqq \|x\|$ for all $x \in X$ and thus $\|x\| \leqq |||x||| \leqq K\|x\|$. Moreover, letting $x_\alpha = z_\alpha / f_\alpha(z_\alpha)$, the system $\{x_\alpha, f_\alpha : \alpha \in A\}$ satisfies the hypotheses of Lemma 1, using the norm $|||\,.\,|||$, with

$$\lambda = \max(1+D, 1+M+rD)/(Ms+D-1) < 1.$$

This completes the proof.

A Banach space is *uniformly convex* (*uniformly rotund*) (see Day [3]) if

$$\delta(\varepsilon) = \inf\{1 - \|x+y\|/2 : \|x\| \leqq 1, \|y\| \leqq 1, \|x-y\| \geqq \varepsilon\}$$

is positive for all $0 < \varepsilon \leqq 2$. A space is *super-reflexive* when it has an equivalent uniformly convex norm. In this case Theorem 1 may be strengthened to show that super-reflexive spaces are arbitrarily nearly isometric to spaces with property B, although it is still unknown whether super-reflexive spaces themselves always have the property.

THEOREM 2. *Let $(X, \|.\|)$ be a super-reflexive space. Then for any $K > 1$ there is a norm $|||\,.\,|||$ on X such that $\|x\| \leqq |||x||| \leqq K\|x\|$ and $(X, |||\,.\,|||)$ has property B.*

PROOF. If X is super-reflexive it is arbitrarily nearly isometric to a uniformly convex space, for if $\|x\| \leqq p(x) \leqq C\|x\|$ and p is a uniformly convex norm on X then the norm $q(x) = r\|x\| + (1-r)p(x)$ is uniformly convex for any $0 < r < 1$ and satisfies $\|x\| \leqq q(x) \leqq (r + C(1-r))\|x\|$. We may therefore assume without loss of generality that X is uniformly convex.

Since $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ we may select $\varepsilon > 0$ such that $\min(1-\varepsilon, 1-2\delta(\varepsilon)) \geqq 1/K$. We will write δ to denote $\delta(\varepsilon)$.

Let $(y_\alpha, g_\alpha)_{\alpha<\gamma}$ be a system such that for each α, $y_\alpha \in X$, $g_\alpha \in X^*$, and $\|y_\alpha\| = \|g_\alpha\| = g_\alpha(y_\alpha) = 1$, and such that $\{y_\alpha : \alpha < \gamma\}$ is dense in the unit sphere of X. By transfinite induction we may select a subset A of γ satisfying the conditions $0 \in A$ and $0 < \alpha \in A$ if and only if $|g_\beta(y_\alpha)| \leqq 1 - 2\delta$ and $|g_\alpha(y_\beta)| \leqq 1 - 2\delta$ for all $\beta \in A$ with $\beta < \alpha$. Let $f_\alpha = Kg_\alpha$ and $x_\alpha = y_\alpha / K$ for $\alpha \in A$.

It follows that $|||x||| = \sup_{\alpha \in A} |f_\alpha(x)| \leqq K\|x\|$ for all $x \in X$. If $\alpha \in A$, $|||x_\alpha||| = 1$; if $\alpha \notin A$ there is a $\beta \in A$, $\beta < \alpha$, such that either

(i) $|f_\beta(x_\alpha)| > 1 - 2\delta$, or

(ii) $|f_\alpha(x_\beta)| > 1 - 2\delta$.

In the latter case, $f_\alpha(x_\alpha + \theta x_\beta) > 2 - 2\delta$ for some θ of modulus one, and so $\|x_\alpha + \theta x_\beta\| > (2 - 2\delta)/K$, and thus $\|x_\alpha - \theta x_\beta\| \leqq \varepsilon/K$, and so $|f_\beta(x_\alpha - \theta x_\beta)| \leqq \varepsilon$ and $|f_\beta(x_\alpha)| \geqq 1 - \varepsilon$.

In each case $|f_\beta(y_\alpha)| \geqq 1$ and hence $|||\,x\,||| \geqq \|x\|$ for all x since $\{y_\alpha : \alpha < \gamma\}$ is dense in the unit sphere of X. Moreover the system $\{x_\alpha, f_\alpha : \alpha \in A\}$ satisfies the hypotheses of Lemma 1, using the norm $|||\,.\,|||$, with $\lambda = 1 - 2\delta$. The result follows.

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